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A multifield model for blocky materials based on multiscale description

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Abstract

A multiscale approach to model the mechanical behaviour of blocky materials that exhibit microscale features is proposed. From the description of these materials at microscopic level, as systems of interacting rigid elements, a formula for the stored energy is given in order to derive the macroscopic constitutive equations of a linear elastic equivalent multifield continuum. This continuum results to be a micropolar continuum suitable to describe the mechanical behaviour of brick/block masonry, as well as jointed rocks or matrix/particle composites, accounting for the size, the orientation and the arrangement of the elements. This multiscale approach proves to be effective also in the non-linear framework. The material non-linear behaviour is represented through internal constraints derived from delimitations imposed to the interactions of the block system. Some numerical examples show the correspondence between discrete and continuum modelling, both in the linear and in the non-linear frame.

Keywords: Continua with microstructure; Multifield continua; Micropolar continua; Multiscale models; Molecular elasticity

1. Introduction

The continuous development of structural analyses' tools concerning the behaviour of masonry-like materials, in order to preserve historical manufacts and to study rocks mechanics problems, requires the definition of models for gross behaviour of materials which show, at finer scales, heterogeneities of significant size and texture. Analogous problems concern the mechanics of complex materials, characterized by the presence of different kind of heterogeneities like lattice defects (rigid or soft inclusions, voids, microcracks, etc.) (Nemat-Nasser and Hori, 1993) or undergoing phase transitions (Fischer et al., 1994).

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A fundamental problem in the formulation of such models is the identification of suitable constitutive laws capable to describe discontinuous or heterogeneous materials by means of homogeneous continua, filtering enough information to provide a global description of the mechanical behaviour. Although the homogenisation techniques have been considerably improved since long time (Hashin, 1983; Sanchez-Palencia, 1987), the models proposed so far in this framework are generally based on classical continua and exhibit some drawbacks (Anthoine, 1995; de Felice, 1995; Cecchi and Sab, 2002). For example, due to the absence of scale parameters and appropriate kinematical descriptors, the use of a simple Cauchy model ("of degree 1") makes impossible to distinguish the behaviour of media made of particles of different size and orientation. This model turns out to be not suited to study, even in elastic regime, problems with geometric or loading singularities that involve high stress and strain gradients (Trovalusci and Masiani, 1999, 2003). The main difficulties arise when the characteristic size of the body under consideration is of the same order of magnitude of the size of the internal heterogeneities. The problem is further complicated when non-linear and non-monotone stress–strain laws are required. In this case, ill-positioning of the field equations as well as strongly mesh-dependent finite element solutions may arise (Read and Hegemier, 1984; Simo, 1989; Sluys et al., 1993).

Enriched Cauchy continua are often employed in order to circumvent the above mentioned ill-conditionings (Pijaudier-Cabot and Bažant, 1987; Needleman, 1988; Mühlhaus and Aifantis, 1991; Pijaudier-Cabot, 2000), but the presence of non-standard strain measures (implying spatial or time derivatives of order different than the second in the equations of motion) without corresponding stress measures leads to thermodynamic inconsistency (Gurtin, 1965), unless resorting to the introduction of, more or less artificial, dissipation inequalities. Therefore it becomes necessary to define different kinds of models free from such limitations but still capable to provide a description in terms of continuous media for materials whose internal structure influences substantially the macroscopic behaviour (Mühlhaus, 1995; Eringen, 2001).

To this end, the so-called "multifield theory" provides a sound solution to the above mentioned difficulties concerning the macroscale description of the complex materials with significant microscopic features (Capriz, 1989; Mariano, 2001). The models developed in this framework have to be understood as continua with different material levels: the 'macrostructural' level of the matrix and one or more 'microstructural' levels, characterized by the presence of descriptors additional to the standard ones. Therefore, non-standard strain and stress measures can be defined in a rational way, suitable scale parameters can be naturally introduced and it can be easily shown that the thermodynamic consistency is ensured.

The applicability of such models relies on the possibility to define constitutive functions for all the stress measures introduced. This problem has been studied for some particular microstructures—materials with continuous distributions of cracks, composite materials of short-fibre type, masonry materials (Trovalusci and Augusti, 1998; Mariano and Trovalusci, 1999; Trovalusci and Masiani, 1999, 2003; Trovalusci, 2003)—in an alternative way with respect to the classical homogenisation procedures. In these works an integral procedure of equivalence to govern the transition between the micro and the macroscale has been proposed. This procedure is based on the requirement that a micro and a macro model of a given material spent the same mechanical power in corresponding velocity fields and, differently from the homogenisation techniques, does not require the solution of a boundary problem on a representative volume element. Using this approach however, additional criteria to select the suitable multifield continuum are required.

A basic issue when dealing with continuum homogeneous modelling of heterogeneous and/or discontinuous materials is the selection of the most appropriate continuous model to grossly describe their behaviour taking into account the features exhibited at finer scales. Although it not always obvious, especially in the presence of different kind of microstructures (inclusions, voids, interfaces, etc.), this choice is generally made 'a priori' on the basis of phenomenological issues. In the work (Trovalusci and Masiani, 1999) this question has been faced and, in order to make this choice more objective, a criterion—based on the preservation of the materials symmetries in the transition from the macro to the microscale—has been defined.[1]

In this work an energy equivalence procedure based on a multiscale approach is proposed. From the description of the material as a lattice model, made of interacting micro-heterogeneities ('micromodel'), a formula for the strain energy is given in order to derive the macroscopic constitutive equations of an equivalent multifield continuous model ('macromodel'). The macromodel grossly represents the material mechanical behaviour, not only taking into account the shape and the disposition of the internal elements but also their size and orientation. This approach, is conceptually analogous to the power equivalence approach but it eliminates the necessity of additional criteria to select the proper macromodel. It is shown that when the micromodel is adopted and the class of its admissible deformations selected, the macromodel is defined. In this case the mechanical powers and the material symmetries of the two models naturally correspond. This is similar to what proposed in the molecular theory of elasticity, which also aims to derive macroscopic constitutive equations from the study of systems of material particles ('molecules') interacting in various ways (Ericksen, 1977). Analogous approaches have been generally used in crystal lattice elasticity (Askar, 1985; Ortiz and Phillips, 1999, cap. IV) and they still appear potential of making the multifield theories motivated from a constitutive point of view. Moreover, based on non-quadratic energies of lattice systems, non-simple constitutive models can be derived suitable of describing complex materials accounting for the presence of defects (Mura, 1987; Gallego and Ortiz, 1993).

In particular, attention is here focused on the constitutive modelling of masonry-like materials. For such materials the original lattice model is made of rigid particles interacting in pair through forces and couples, both having linear and non-linear response functions. It is shown that by assuming homogeneous deformations for the micromodel, accounting for simplicity only for short-range interactions, it is possible to give an expression for the strain energy from which deriving the constitutive functions for all the stress measures of the equivalent continuum. In this case the corresponding continuum results to be a continuum with rigid local structure (micropolar continuum), whose linearised kinematical descriptors are the vector field of the standard displacement and the skew-symmetric tensor field of the microrotation. Moreover, in order to take into account the material non-linear behaviour, internal constraints for the macromodel are derived from the knowledge of the failure conditions on the micromodel. The possibility to introduce constraints for the strain energy, considering local discrepancies in the micromodel, is one of the main advantages of the energy the multiscale approach, that allows avoiding the definition of evolution laws and yield functions on the macromodel and the snags to the operation related (Besdo, 1985; Mühlhaus, 1989; de Buhan et al., 2002).

Some numerical analyses are performed on two-dimensional masonry assemblages using a specific finite element code, which supply linear and non-linear solutions for the multifield model. In order to show the effectiveness of the micropolar model to describe blocky materials, the results are compared with the results obtained using codes based on discrete modelling of such materials (Trovalusci, 1992; Baggio and Trovalusci, 2000). In particular, it is shown how the presence of the microrotation field accounts for the relative rotation among rigid elements with adequate accuracy. Such rotation, as shown experimentally, turns out to be determinant in recognizing the collapse mechanisms of masonry structures.

2. The lattice (micro) model

In order to derive the elastic constitutive relationships for a continuum that can represent the macroscopic behaviour of masonry-like materials, a multiscale approach is adopted. The first step is the descrip-

[1] In particular, it has been shown that adopted a micromodel for a material the power equivalent macromodel that preserves the same material symmetries is unique. In absence of such a criterion the equivalence procedure allows to identify any equivalent model on condition than some terms were cut off, although they could play an important role (as for instance the mutual particles rotations when a Cauchy model is adopted to describe rigid particles materials).

tion of the material, at the microscopic level, as a system of rigid elements interacting in pair through forces and couples. This hypothesis accounts for the higher deformability of the joints with respect to the deformability of blocks. In the following we refer to this lattice model as 'micromodel'.

In a linearised framework, the kinematics of the micromodel is described by two discrete fields: the vector field of the displacement of the centre, $\mathbf{a}$, of the generic element A, $\mathbf{u}^a$, and the skew-symmetric tensor field of the (rigid) rotation of A, $\mathbf{W}^a$. Considering two interacting elements, A and B, the strain measures of the system are the relative displacement, $\mathbf{u}_p$, and the relative rotation, $\mathbf{W}_p$, between two point, $\mathbf{p}^a$ and $\mathbf{p}^b$, respectively on A and B,

$$\begin{aligned} \mathbf{u}_p &= \mathbf{u}^a - \mathbf{u}^b + \mathbf{W}^a(\mathbf{p}^a - \mathbf{a}) - \mathbf{W}^b(\mathbf{p}^b - \mathbf{b}), \\ \mathbf{W}_p &= \mathbf{W}^a - \mathbf{W}^b. \end{aligned} \tag{1}$$

Denoted with $\mathbf{f}^a$ and $\mathbf{M}^a$ the vector and the skew-symmetric tensor of the external forces and couples acting on A respectively, the balance equations for the assembly write

$$\begin{aligned} &\sum_{p=1}^{N_p^a}(\mathbf{t}_p^a) + \mathbf{f}^a = \mathbf{0} \\ &\sum_{p=1}^{N_p^a}\mathbf{C}_p^a - [\mathbf{t}_p^b \otimes (\mathbf{p}^a - \mathbf{p}^b) - (\mathbf{p}^a - \mathbf{p}^b) \otimes \mathbf{t}_p^b] + \mathbf{M}^a = \mathbf{0}, \end{aligned} \tag{2}$$

for each element A of the system with

$$\begin{aligned} &\mathbf{t}_p^a + \mathbf{t}_p^b = \mathbf{0} \\ &\mathbf{C}_p^a + \mathbf{C}_p^b - [\mathbf{t}_p^b \otimes (\mathbf{p}^a - \mathbf{p}^b) - (\mathbf{p}^a - \mathbf{p}^b) \otimes \mathbf{t}_p^b] = \mathbf{0} \end{aligned} \tag{3}$$

for each pair of interacting elements A and B. In the above equations N_p^a is the number of elements interacting with A, $\mathbf{t}_p^a$ and $\mathbf{C}_p^a$ are the force and couple that each adjacent element B exerts on A, respectively.

Considering the material as periodic, or at least statistically homogeneous, a representative volume element, referred to as the 'module', can be found. From the balance Eqs. (2) and (3) the formula for the mean internal power over the volume, V, of the module can be derived

$$\bar{\pi}(\dot{\mathbf{u}}_p, \dot{\mathbf{W}}_p) = \frac{1}{V}\left\{\sum_p\left(\mathbf{t}_p \cdot \dot{\mathbf{u}}_p + \frac{1}{2}\mathbf{C}_p \cdot \dot{\mathbf{W}}_p\right)\right\}, \tag{4}$$

where the summation is extended to each pair of interacting elements of the module.

Assuming as response functions for the interactions between the pairs of elements, $\mathbf{t}_p = \mathbf{t}_p^a$ and $\mathbf{C}_p = \mathbf{C}_p^a$, the linear elastic functions

$$\begin{aligned} \mathbf{t}_p &= \mathbf{K}_p\mathbf{u}_p, \\ \mathbf{C}_p &= \mathsf{K}_p\mathbf{W}_p, \end{aligned} \tag{5}$$

where $\mathbf{K}_p$ and K_p are symmetric constitutive tensors of the second and fourth order, respectively, the mean strain energy function over the module can be written as a quadratic form

$$\bar{\varepsilon}(\mathbf{u}_p, \mathbf{W}_p) = \frac{1}{2V}\sum_p\left(\mathbf{K}_p\mathbf{u}_p \cdot \mathbf{u}_p + \frac{1}{2}\mathsf{K}_p\mathbf{W}_p \cdot \mathbf{W}_p\right). \tag{6}$$

In many circumstances however, the behaviour of blocky materials is strongly influenced by the material non-linearity, such as the low tensile strength and the frictional proprieties. Attention is focused on materials, like ancient masonry or jointed rock assemblages, made of dry-stacked elements or with joints filled by poor and scattered mortar, for which the lack of coherence and the frictional strength can be supposed concentrated along the contact surfaces between blocks. In order to derive the constitutive relations for the micropolar continuum also in the non-linear frame, tacking into account these features, delimitations on the contact actions of the discrete assembly must be posed.

Let $\{\mathbf{m}_{p1}, \mathbf{m}_{p2}, \mathbf{n}_p\}$ be an orthonormal basis for the contact surface between the pth pair of blocks, where $\mathbf{m}_{p1}$, $\mathbf{m}_{p2}$ and $\mathbf{n}_p$ are respectively two unit vectors tangent and one unit vector normal to the contact surface. Considering plane contact surfaces, discretised with L_p pairs of parallel linear edges defined by their unit normals, $\mathbf{m}_{p\alpha}$, with $\mathbf{m}_{p\alpha} \cdot \mathbf{n}_p = 0$ and $\alpha = 1, L_p$, the constitutive functions for the contact actions, $\mathbf{t}_p$ and $\mathbf{C}_p$, of each contact surface p of the module (4) are subjected to the following restrictions

$$
\begin{aligned}
&\left\|(\mathbf{I} - \mathbf{n}_p \otimes \mathbf{n}_p)\mathbf{t}_p\right\| \leqslant \tan\phi_p(a_p - \mathbf{t}_p \cdot \mathbf{n}_p),\\
&\mathbf{t}_p \cdot \mathbf{n}_p \leqslant a_p,\\
&\left|\mathbf{C}_p\mathbf{m}_{p\alpha} \cdot \mathbf{n}_p\right| \leqslant d_p^{\alpha}(a_p - \mathbf{t}_p \cdot \mathbf{n}_p), \quad \alpha = 1, L_p,\\
&\left|\mathbf{C}_p\mathbf{m}_{p1} \cdot \mathbf{m}_{p2}\right| \leqslant \tan\phi_p d_p \beta_p(a_p - \mathbf{t}_p \cdot \mathbf{n}_p),
\end{aligned}
\tag{7}
$$

where a_p is the tensile strength of the material interposed between blocks; ϕ_p is the friction angle, d_p and d_p^{α} are characteristic lengths and β_p is a non-dimensional factor accounting for the shape of the pth contact surface; $(\mathbf{I} - \mathbf{n}_p \otimes \mathbf{n}_p)$ is the projection tensor on the pth surface. The first and the last of the above inequalities ensure that the tangential components of the contact force and the torsional component of the contact couple, respectively, do not exceed the Mohr-Coulomb frictional strength. The second inequality makes sure that the normal component of the contact force does not overcome the tension strength of the material interposed between blocks. The third set of inequalities guarantees that the normal component of the contact force is applied within the contact surface; these inequalities, if the contact couple is zero, imply the second one.

The above delimitations characterise a discrete system made of rigid elements, supposed to have infinite strength, in contact through surfaces, with bounded tensile strength and resistant to sliding by friction. In a continuum smeared conception, the assumption that blocks cannot break does not appear too restrictive especially if, as in this case, the geometry of the assembly, rather than the strength of the units, captures the actual ultimate behaviour of masonry materials. Anyway, it would not be difficult to consider systems of breakable elements.

3. The linear-elastic multifield (macro) model

Multifield models are continua capable to model complex media with different kinds of microstructure-inclusions (rigid or soft), cracks, voids, interfaces between solid phases, etc.—retaining memory of the material's fine organization at different scales. The basic starting point is to consider the generic material patch as a system and to introduce, already at the geometrical level of the description of the body, information on the material microstructure. The material patch is characterized not only by its spatial position (as in the mechanics of simple materials) but also by suitable descriptors of the microstructural morphology, whose rate of change is associated to interactions that satisfy suitable balance equations and pose non-trivial constitutive problems.

In order to represent the actual macroscopic behaviour of complex materials, a crucial step is the selection of the most appropriate multifield continuum for the description of the specific internal structure exhibited at the considered microscale.

In this work the uncertainty about the definition of the continuum is by-passed following the approach proposed in the molecular theory of elasticity (Ericksen, 1977), which aims at predicting macroscopic constitutive functions of continua equivalent to lattice systems. Whenever the original lattice model has been defined, by selecting the admissible deformation fields for the micro model, it is possible to give an expression for the strain energy from which the constitutive functions for the stress measures of the resulting equivalent continuum can be derived.

As usual in the molecular theory of elasticity, it is assumed that the elements of the lattice system described in Section 2 are subjected to homogeneous deformations in such a way that the linearised kinematical fields have the affine representation

$$\begin{aligned} \mathbf{u}^a &= \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{a}-\mathbf{x}),\\ \mathbf{W}^a &= \mathbf{W}(\mathbf{x}) + \nabla\mathbf{W}(\mathbf{a}-\mathbf{x}), \end{aligned} \tag{8}$$

with $\mathbf{u}$ and $\mathbf{W}$ a vector and a skew-symmetric tensor field, respectively. This hypothesis has intuitive interpretation and can be considered valid if short-range interactions are accounted for. On the basis of Eqs. (8), the strain measures (1) of the lattice system can be expressed in terms of continuum fields

$$\begin{aligned} \mathbf{u}_p(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W}) &= (\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b}) + \nabla\mathbf{W}(\mathbf{a}-\mathbf{x})(\mathbf{p}^a-\mathbf{a}) - \nabla\mathbf{W}(\mathbf{b}-\mathbf{x})(\mathbf{p}^b-\mathbf{b}),\\ \mathbf{W}_p(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W}) &= \nabla\mathbf{W}(\mathbf{a}-\mathbf{b}) \end{aligned} \tag{9}$$

as well as the mean strain energy function over the module

$$\begin{aligned} \bar{\varepsilon}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W}) = \frac{1}{2V}\Bigg\{ & (\nabla\mathbf{u}-\mathbf{W})\cdot\sum_p \mathbf{K}_p[(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b}) + \nabla\mathbf{W}(\mathbf{a}-\mathbf{x})(\mathbf{p}^a-\mathbf{a})\\ & - \nabla\mathbf{W}(\mathbf{b}-\mathbf{x})(\mathbf{p}^b-\mathbf{b})]\otimes(\mathbf{a}-\mathbf{b}) + \nabla\mathbf{W}\cdot\Bigg[\sum_p \mathbf{K}_p[(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b}) + \nabla\mathbf{W}(\mathbf{a}-\mathbf{x})(\mathbf{p}^a-\mathbf{a})\\ & - \nabla\mathbf{W}(\mathbf{b}-\mathbf{x})(\mathbf{p}^b-\mathbf{b})]\otimes[(\mathbf{p}^a-\mathbf{a})\otimes(\mathbf{a}-\mathbf{x}) - (\mathbf{p}^b-\mathbf{b})\otimes(\mathbf{b}-\mathbf{x})] + \frac{1}{2}\sum_p \mathsf{K}_p\nabla\mathbf{W}(\mathbf{a}-\mathbf{b})\otimes(\mathbf{a}-\mathbf{b})\Bigg]\Bigg\}. \end{aligned} \tag{10}$$

Let now consider a closed ball N, of radius δ centred at $\mathbf{x}$ and of volume $V(N_\delta)$, of the regular Euclidean region occupied by the continuum. It is assumed that a stored energy function exists and, through the localization theorem, coincides with the mean strain energy of the module

$$\lim_{\delta\to 0}\frac{1}{V(N_\delta)}\int_{N^\delta}\varepsilon\,\mathrm{d}V := \bar{\varepsilon}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W}). \tag{11}$$

Once derived the expression for the strain energy density, the type of the microstructure of the continuum is sought. In this case, the stress measures of the continuum are identified as

$$\begin{aligned} \mathbf{S}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W}) &= \frac{\partial\bar{\varepsilon}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W})}{\partial(\nabla\mathbf{u}-\mathbf{W})}\\ &= \frac{1}{V}\sum_p\left\{\mathbf{K}_p[(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b}) + \nabla\mathbf{W}(\mathbf{a}-\mathbf{x})(\mathbf{p}^a-\mathbf{a}) - \nabla\mathbf{W}(\mathbf{b}-\mathbf{x})(\mathbf{p}^b-\mathbf{b})]\otimes(\mathbf{a}-\mathbf{b})\right\},\\ \mathsf{S}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W}) &= \frac{2\partial\bar{\varepsilon}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W})}{\partial(\nabla\mathbf{W})}\\ &= \frac{2}{V}\sum_p\Bigg\{\mathbf{K}_p[(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b}) + \nabla\mathbf{W}(\mathbf{a}-\mathbf{x})(\mathbf{p}^a-\mathbf{a}) - \nabla\mathbf{W}(\mathbf{b}-\mathbf{x})(\mathbf{p}^b-\mathbf{b})]\otimes[(\mathbf{p}^a-\mathbf{a})\otimes(\mathbf{a}-\mathbf{x})\\ &\quad -(\mathbf{p}^b-\mathbf{b})\otimes(\mathbf{b}-\mathbf{x})] + \frac{1}{2}\mathsf{K}_p\nabla\mathbf{W}(\mathbf{a}-\mathbf{b})\otimes(\mathbf{a}-\mathbf{b})\Bigg\}. \end{aligned} \tag{12}$$

After some algebraic manipulations, the constitutive functions for the stress measures can be written as

$$\begin{aligned}&\mathsf{S} = \mathsf{A}(\nabla \mathbf{u} - \mathbf{W}) + \mathsf{B}\nabla \mathbf{W},\\ &\mathsf{S} = \mathsf{B}^{\mathrm{T}}(\nabla \mathbf{u} - \mathbf{W}) + \mathsf{C}\nabla \mathbf{W},\end{aligned} \tag{13}$$

where the elastic tensors A, B, and C, of the fourth, fifth and sixth order, respectively, have components depending on the elastic constants of the matrix and on the shape, the size, the orientation and the arrangement of the elements of the lattice system.[2] In particular, the tensor A does not contain any internal length parameter while the tensors B and C have components depending also on the size of the elements. These components are[3]

$$\begin{aligned}
(\mathsf{A})_{ijhk} &= \frac{\partial^2\bar{\varepsilon}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W})}{\partial(\nabla\mathbf{u}-\mathbf{W})_{ij}\partial(\nabla\mathbf{u}-\mathbf{W})_{hk}} = \frac{1}{V}\sum_p\{(\mathbf{K}_p)_{ih}(\mathbf{a}-\mathbf{b})_k(\mathbf{a}-\mathbf{b})_j\},\\
(\mathsf{B})_{ijhkl} &= \frac{\partial^2\bar{\varepsilon}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W})}{\partial(\nabla\mathbf{u}-\mathbf{W})_{ij}\partial(\mathbf{W})_{hkl}} = \frac{1}{V}\sum_p\{(\mathbf{K}_p)_{ih}[(\mathbf{a}-\mathbf{x})_l(\mathbf{p}^a-\mathbf{a})_k-(\mathbf{b}-\mathbf{x})_l(\mathbf{p}^b-\mathbf{b})_k](\mathbf{a}-\mathbf{b})_j\},\\
(\mathsf{C})_{hklmnq} &= \frac{2\partial^2\bar{\varepsilon}(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W})}{\partial(\nabla\mathbf{W})_{hkl}\partial(\nabla\mathbf{W})_{mnq}}\\
&= \frac{2}{V}\sum_p\Big\{(\mathbf{K}_p)_{hm}[(\mathbf{a}-\mathbf{x})_q(\mathbf{p}^a-\mathbf{a})_n(\mathbf{p}^a-\mathbf{a})_k(\mathbf{a}-\mathbf{x})_l-(\mathbf{b}-\mathbf{x})_q(\mathbf{p}^b-\mathbf{b})_n(\mathbf{p}^b-\mathbf{b})_k(\mathbf{b}-\mathbf{x})_l]\\
&\quad+\frac{1}{2}(\mathsf{K}_p)_{hkmn}(\mathbf{a}-\mathbf{b})_q(\mathbf{a}-\mathbf{b})_l]\Big\}.
\end{aligned} \tag{14}$$

The major symmetries hold in such a way that $\mathsf{A}_{ijhk} = \mathsf{A}_{hkij}$, $\mathsf{B}_{ijhkl} = \mathsf{B}_{hklij}$ and $\mathsf{C}_{hklmnq} = \mathsf{C}_{mnqhkl}$. Moreover, if the material is centro-symmetric[4], the tensor B is null and equations (13) are uncoupled.

The stress measures identified correspond to the stress measures of an anisotropic micropolar continuum. In a linearised framework the kinematics of the multifield continuum is described by two fields: the displacement vector, $\mathbf{u}$, and the microrotation tensor, $\mathbf{W}$. The linearised strain measures are the second order strain tensor, $\nabla\mathbf{u}-\mathbf{W}$, and the so-called curvature tensor, $\nabla\mathbf{W}$, of the third order. Denoted with $\mathbf{b}$ and $\mathbf{B}$ the density of body force and couple, respectively, from balance equations

$$\begin{aligned}&\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0},\\ &\operatorname{div}\mathsf{S} - (\mathbf{S} - \mathbf{S}^T) + \mathbf{B} = \mathbf{0}\end{aligned} \tag{15}$$

the power density formula of this continuum is

$$\pi(\dot{\mathbf{u}}, \dot{\mathbf{W}}) = \mathbf{S}\cdot(\nabla\dot{\mathbf{u}} - \dot{\mathbf{W}}) + \frac{1}{2}\mathsf{S}\cdot\nabla\dot{\mathbf{W}}. \tag{16}$$

[2] B^{T} is the transpose of the tensor B such that, for each second order tensor $\mathbf{A}$ and each third order tensor A, $\mathsf{B}\mathbf{A}\cdot\mathsf{A} = \mathsf{B}^{\mathrm{T}}\mathsf{A}\cdot\mathbf{A}$.

[3] Assuming an orthonormal basis $\{\mathbf{e}_i\}$, $i = 1,n$, the components of a tensor T of order n are

$$(\mathsf{T})_{i_1 i_2 \cdots . i_n} = \mathsf{T}\cdot\mathbf{e}_{i_1}\otimes\mathbf{e}_{i_2}\otimes\cdots\otimes\mathbf{e}_{i_n}.$$

[4] The central symmetry is the material symmetry of any periodic assemblages of elements.

Using Eqs. (5) and (8) and considering the found constitutive equations (12) it can be recognized that the density power of this micropolar continuum equals the mean power over the module of the lattice system (4). Moreover, It can be easily shown that and that the material symmetries of the two models also correspond.[5]

The identification of a micropolar continuum is a direct consequence of the hypothesis (8). If a different class of admissible deformations for the lattice system is selected the stress measures of a different continuum can be identified. For example, supposing the lattice system made of particles in contact ($\mathbf{p}^a - \mathbf{p}^b = \mathbf{0}$) that cannot rotate one each other and then assuming as class of admissible deformations

$$\begin{aligned} \mathbf{u}^a &= \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{a} - \mathbf{x}), \\ \mathbf{W}^a &= \mathbf{W}(\mathbf{x}), \end{aligned} \tag{17}$$

with $\mathbf{W} = (\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}})$, it can be shown that a Cauchy equivalent continuum can be identified.[6] It is worth's noting that the classical equivalent continuum cannot be obtained from the micropolar continuum only by vanishing the size of the particles in the lattice system. It has been shown (Trovalusci and Masiani, 1999) that when the length parameter goes to zero, B and C vanish but A does not coincide with the Cauchy elastic tensor, unless the material belongs at least to the class of the orthotetragonal materials, like isotropic materials.

3.1. Numerical examples

To investigate the effectiveness of the micropolar equivalent continuum in modelling the gross behaviour of discontinuous block systems, some analyses were performed on two-dimensional block structures. The unit vectors $\mathbf{e}_1$ (horizontal direction) and $\mathbf{e}_2$ (vertical direction) represent an orthonormal basis for these structures. The walls examined are made of orthotropic texture of rectangular blocks, of size (4×2) 10^{-1} m, along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively.

The numerical solution for the Cosserat continuum was evaluated by means of a F.E. discretised model. A specific three-node triangular element was formulated. This element has three degrees of freedom per node: two in-plane translations and one in-plane rotation. The two components of displacement and the rotations are assumed linear. All the corresponding strain measures are constant except the non-symmetric strains, $(\nabla\mathbf{u} - \mathbf{W})_{12} = (\nabla\mathbf{u} - \mathbf{W})\mathbf{e}_1 \otimes \mathbf{e}_2$ and $(\nabla\mathbf{u} - \mathbf{W})_{21} = (\nabla\mathbf{u} - \mathbf{W})\mathbf{e}_2 \otimes \mathbf{e}_1$, that are linear.

In order to make a comparison, the solution for the lattice model was also directly obtained. The algorithm used to obtain the numerical solution is based on a model made of rigid blocks, described by 'constraint equations', interacting between interfaces, described by longitudinal, transversal and rotational springs (Trovalusci, 1992).

As first example a wall of dimensions $(8 \times 8)m$, fixed on the basis and loaded by a constant field of body forces whose density has the vertical and the horizontal components in the ratio 2:1, was analysed. Considering the orthonormal local basis $\{\mathbf{m}_p, \mathbf{n}_p\}$ for the pth contact surface between blocks, where $\mathbf{m}_p$, and $\mathbf{n}_p$ are respectively the unit vectors tangent and normal to the contact surface, the non-null constitutive constants for the discrete model are:

[5] The constitutive constants obtained (14) coincide, in the two-dimensional frame, with the ones obtained in the work (Trovalusci and Masiani, 1996). In this work the elastic constant of a Cauchy continuum were also derived and they coincide with the ones obtained using a classical homogenisation procedure (de Felice, 1995).

[6] The elastic constant of the equivalent Cauchy continuum coincide with the ones obtained using a classical homogenisation procedure (de Felice, 1995).

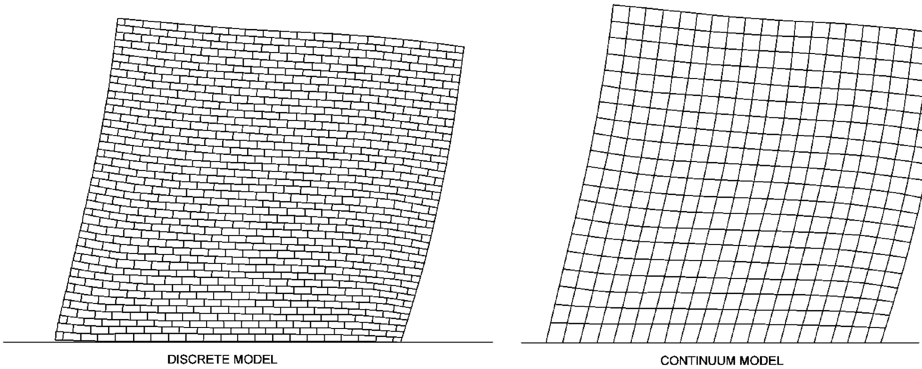

Fig. 1. Wall with body forces. Deformed shapes.

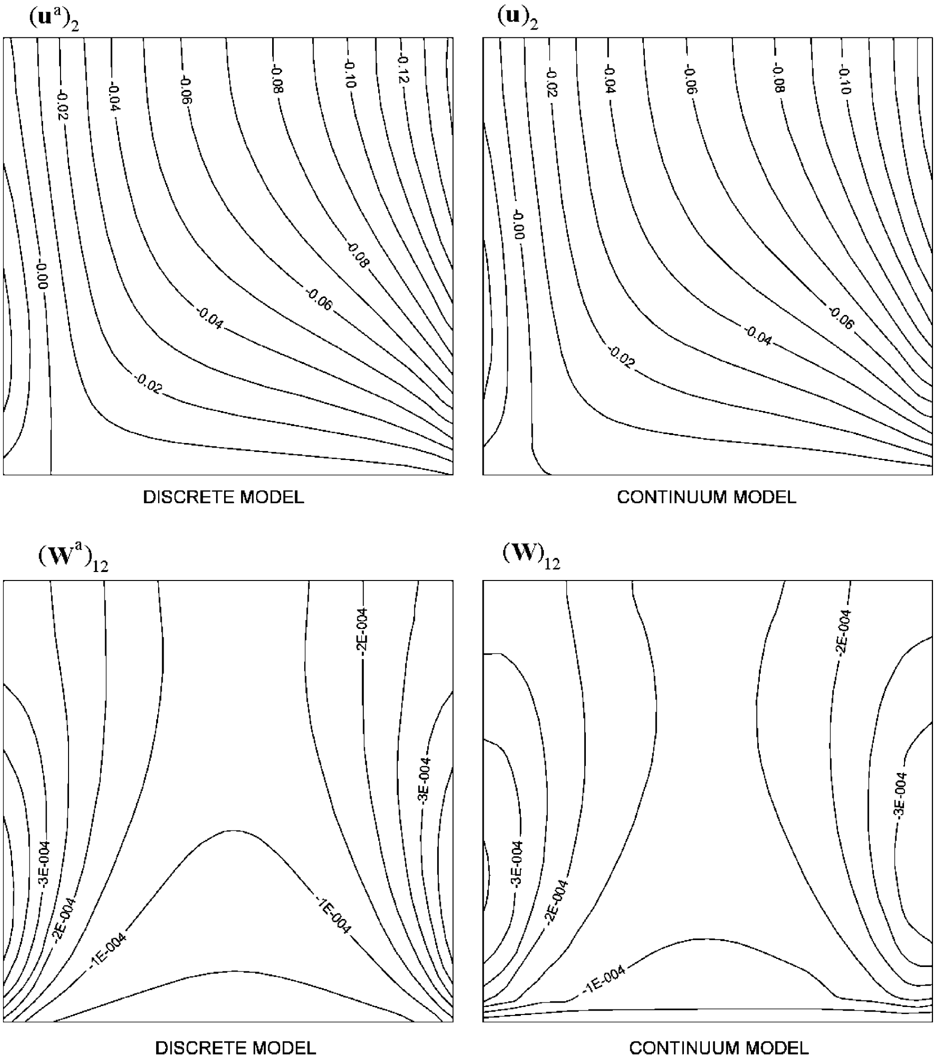

Fig. 2. Wall with body forces. Contour lines of the vertical component of the displacement fields (top) and of the local rotation fields (bottom).

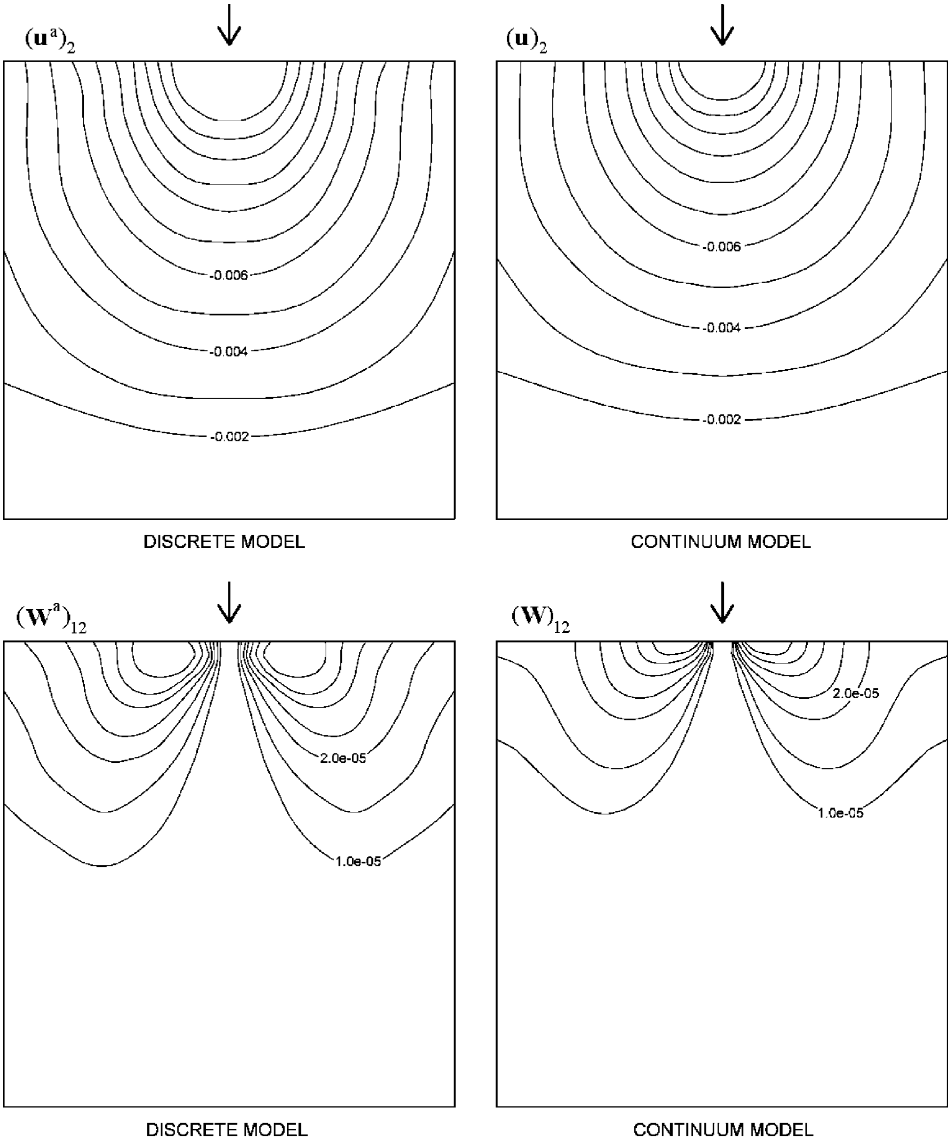

Fig. 3. Wall with concentrated load. Contour lines of the vertical component of the displacement fields (top) and of the local rotation fields (bottom).

$$(\mathbf{K}_p)_{mm} = \mathbf{K}_p \cdot \mathbf{m}_p \otimes \mathbf{m}_p = 1.00 \cdot 10^3 \text{ MN/m}; (\mathbf{K}_p)_{nn} = \mathbf{K}_p \cdot \mathbf{n}_p \otimes \mathbf{n}_p = 5.00 \cdot 10^3 \text{ MN/m},$$
$$(\mathsf{K}_p)_{mnmn} = \mathsf{K}_p \cdot \mathbf{m}_p \otimes \mathbf{n}_p \otimes \mathbf{m}_p \otimes \mathbf{n}_p = 5.00 \cdot 10 \text{ MNm}.$$

The non-null elastic constants derived for the orthotropic micropolar continuum using equations (14) are:

$$(\mathsf{A})_{1111} = \mathsf{A} \cdot \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 = 14.67 \cdot 10^3 \text{ MN/m}^2,$$

$$(\mathsf{A})_{2222} = \mathsf{A} \cdot \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 = 6.67 \cdot 10^3 \text{ MN/m}^2,$$

$$(\mathsf{A})_{1212} = \mathsf{A}\cdot\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_2 = 1.34\cdot 10^3\ \mathrm{MN/m^2},$$

$$(\mathsf{A})_{2121} = \mathsf{A}\cdot\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_1 = 9.34\cdot 10^3\ \mathrm{MN/m^2},$$

$$(\mathsf{C})_{121121} = \mathsf{C}\cdot\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_1 = 2.00\cdot 10^2\ \mathrm{MN},$$

$$(\mathsf{C})_{122122} = \mathsf{C}\cdot\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_2 = 0.67\cdot 10^2\ \mathrm{MN}.$$

In Fig. 1 the deformed shapes obtained for the discrete and the continuum model are shown.

Fig. 2 shows on the top side the contour lines of the vertical component of the displacement field for the discrete model $(\mathbf{u}^a)_2$, left, and the continuum model $(\mathbf{u})_2$, right. On the bottom side the contour lines of the rotations of the blocks $(\mathbf{W}^a)_{12}$, left, and of the microrotation $(\mathbf{W})_{12}$, right, are shown.

The same wall was analysed, always as a discrete system and as a micropolar continuum, under the action of a concentrated vertical load, of intensity $2.0\cdot 10^{-3}$ MN, on the middle of the top edge. The contour lines of the components of the discrete and the continuum solution are reported in Fig. 3.

All the results obtained for the Cosserat equivalent model are in good agreement with those obtained for the discrete model.

4. The non-linear micropolar continuum

The proposed multiscale energy approach allows to define a way to treat, with enough simplicity and accuracy, also the macroscopic behaviour of complex materials. Internal constraints for the macromodel can be derived from the knowledge of the failure conditions on the micromodel (7), by-passing the not trivial problem of the definition of a yield domain directly on the continuum model (Besdo, 1985; Mühlhaus, 1989) or of the derivation of a generalised strength criterion, solving an auxiliary yield design problem on a representative cell of the micromodel (de Buhan et al., 2002).

In particular, using Eqs. (7) and (8) and considering systems of blocks disposed according to the central symmetry, the restrictions on the lattice model can be written in terms of strain measures of the macromodel as follows

$$\begin{aligned}
&\left\|(\mathbf{I}-\mathbf{n}_p\otimes\mathbf{n}_p)\mathbf{K}_p(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b})\right\| \leqslant \tan\phi_p(a_p-\mathbf{K}_p(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b})\cdot\mathbf{n}_p),\\
&\mathbf{K}_p(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b})\cdot\mathbf{n}_p \leqslant a_p,\\
&\left|\mathsf{K}_p(\nabla\mathbf{W})(\mathbf{a}-\mathbf{b})\mathbf{m}_{p\alpha}\cdot\mathbf{n}_p\right| \leqslant d_p^{\alpha}(a_p-\mathbf{K}_p(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b})\cdot\mathbf{n}_p),\quad \alpha=1,L_p,\\
&\left|\mathsf{K}_p(\nabla\mathbf{W})(\mathbf{a}-\mathbf{b})\mathbf{m}_{p1}\cdot\mathbf{m}_{p2}\right| \leqslant \tan\phi_p d_p\beta_p(a_p-\mathbf{K}_p(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b})\cdot\mathbf{n}_p).
\end{aligned}\tag{18}$$

The strain energy of the continuum centro-symmetric material, supposed to be placed in the regular Euclidean region C, is

$$\varepsilon(\nabla\mathbf{u}-\mathbf{W},\nabla\mathbf{W}) = \frac{1}{2V}\int_C\left\{(\nabla\mathbf{u}-\mathbf{W})\cdot\sum_p\mathbf{K}_p(\nabla\mathbf{u}-\mathbf{W})(\mathbf{a}-\mathbf{b})+\nabla\mathbf{W}\cdot\frac{1}{2}\sum_p\mathsf{K}_p\nabla\mathbf{W}(\mathbf{a}-\mathbf{b})\otimes(\mathbf{a}-\mathbf{b})\right\}.\tag{19}$$

and it is subjected to the restrictions

$$\begin{aligned}
&\left\|[(\mathbf{I}-\mathbf{n}_p\otimes\mathbf{n}_p)\mathbf{K}_p\otimes(\mathbf{a}-\mathbf{b})](\nabla\mathbf{u}-\mathbf{W})\right\| \leqslant \tan\phi_p(a_p-(\nabla\mathbf{u}-\mathbf{W})\cdot\mathbf{K}_p\mathbf{n}_p\otimes(\mathbf{a}-\mathbf{b})),\\
&(\nabla\mathbf{u}-\mathbf{W})\cdot\mathbf{K}_p\mathbf{n}_p\otimes(\mathbf{a}-\mathbf{b}) \leqslant a_p,\\
&\left|\nabla\mathbf{W}\cdot\mathsf{K}_p\mathbf{m}_{p\alpha}\mathbf{n}_p\otimes(\mathbf{a}-\mathbf{b})\right| \leqslant d_p^{\alpha}(a-(\nabla\mathbf{u}-\mathbf{W})\cdot\mathbf{K}_p\mathbf{n}_p\otimes(\mathbf{a}-\mathbf{b})),\quad \alpha=1,L_p,\\
&\left|\nabla\mathbf{W}\cdot\mathsf{K}_p\mathbf{m}_{p1}\mathbf{m}_{p2}\otimes(\mathbf{a}-\mathbf{b})\right| \leqslant \tan\phi_p d_p\beta_p(a_p-(\nabla\mathbf{u}-\mathbf{W})\cdot\mathbf{K}_p\mathbf{n}_p\otimes(\mathbf{a}-\mathbf{b})).
\end{aligned}\tag{20}$$

These inequalities are obtained, after some algebra, from the inequalities (18) accounting for the symmetries of the tensor $\mathbf{K}_p$ and of the major symmetries of the tensor K_p.

The solution of the problem of such non-linear elastic materials could be directly obtained by searching the minimum of the function $\varepsilon(\nabla\mathbf{u} - \mathbf{W}, \nabla\mathbf{W})$ subjected to the constraints (20). This problem could be numerically faced, for instance, by way of a Monte Carlo code (Sansalone et al., 2004) or other optimisation approaches. However, the presence of Coulomb friction gives rise to non-linear and non-convex mathematical programming problems hard to be solved.[7] Otherwise, a classical way to solve the problem is to resort to the theory of plasticity, as generally done in the past to study no-tension materials (Maier, 1990). This general approach requires the definition of evolution laws for the continuum strain, or stress, variables that are not so easy to be found, especially when the macromodel is provided with a microstructure. A way to derive these laws has been indicated in the work (Trovalusci and Masiani, 1997) in which the strain rates of the micropolar continuum, equivalent to a rigid-particle system with non-linear interfaces, were identified by defining the yield domain in the lattice micromodel. Using the macroscopic flow rules, identified in terms of the geometry of the discrete system, the solution of the elasto-plastic problem can be then obtained using standard techniques.

4.1. The algorithm of solution

To solve the continuum problem, avoiding the above mentioned difficulties, this work proposes an alternative approach that exploits the advantages of the molecular origin of the continuum model. The algorithm of solution elaborated consists in a step-by-step procedure, based on finite element technique, that solves the linear-elastic problem and checks the static admissibility directly on the lattice system. In this framework it is easier to modify the constitutive functions of the contact actions in order to take into account the material non-linear behaviour.

The set-up algorithm works for small load steps as follows.

(i) Given a load step, the continuum problem is solved using the linear Finite Element code, mentioned in Subsection 3.1, providing the strain increments, $\Delta(\nabla\mathbf{u} - \mathbf{W}), \Delta(\nabla\mathbf{W})$.

(ii) For each finite element the compatibility of the contact actions is checked through inequalities (20). If the compatibility is not fulfilled the constitutive tensors in the module of the lattice system, $\mathbf{K}_p$ and K_p, are modified by erasing the relevant components. Consequently, the corresponding components of the contact forces and couples are maintained equal to their limit values, as their increments $\Delta\mathbf{t}_p = \mathbf{K}_p\Delta(\nabla\mathbf{u} - \mathbf{W})(\mathbf{a}-\mathbf{b})$ and $\Delta\mathbf{C}_p = \mathsf{K}_p\Delta(\nabla\mathbf{W})(\mathbf{a} - \mathbf{b})$ vanish.
In particular: if the second of the inequalities (20) is not fulfilled (*opening*) it is assumed

$$\mathbf{K}_p = \mathbf{0}, \quad \mathsf{K}_p = \mathbf{0} \Rightarrow \Delta\mathbf{t}_p = \mathbf{0}, \quad \Delta\mathbf{C}_p = \mathbf{0};$$

if the αth inequality of the set of inequalities (20c) is not verified (*rotation*) it is assumed

$$\mathsf{K}_p\mathbf{n}_p\mathbf{m}_{p\alpha} = \mathsf{K}_p\mathbf{m}_{p\alpha}\mathbf{n}_p = \mathbf{0} \Rightarrow \Delta\mathbf{C}_p\mathbf{m}_{p\alpha}\cdot\mathbf{n}_p = -\Delta\mathbf{C}_p\mathbf{m}_{p\alpha}\cdot\mathbf{n}_p = 0, \quad \alpha = 1, L_p;$$

if the inequality (20a) or one of the inequalities (20d) is not verified (*sliding*) it is assumed

$$(\mathbf{I} - \mathbf{n}_p\otimes\mathbf{n}_p)\mathbf{K}_p = 0, \quad \Rightarrow (\mathbf{I} - \mathbf{n}_p\otimes\mathbf{n}_p)\Delta\mathbf{t}_p = \mathbf{0},$$

$$\mathsf{K}_p\mathbf{m}_{p1}\mathbf{m}_{p2} = \mathsf{K}_p\mathbf{m}_{p2}\mathbf{m}_{p1} = \mathbf{0}, \quad \Rightarrow \Delta\mathbf{C}_p\mathbf{m}_{p2}\cdot\mathbf{m}_{p1} = -\Delta\mathbf{C}_p\mathbf{m}_{p1}\cdot\mathbf{m}_{p2} = 0.$$

[7] The problem of non-linear and non-convex mathematical programming for discrete systems of rigid blocks with no-tension and frictional constraints has been faced in the work (Baggio and Trovalusci, 2000).

(iii) Using the obtained expressions (14), the components of the constitutive tensors of each finite element are modified to take into account the "damage" defined at point (ii) and then, with the usual techniques, a new local stiffness matrix is evaluated.
(iv) The load is increased and the algorithm goes back to point (i).

The cycle stops at the end of the load path or when it is no longer possible to find a balanced solution to the field problem.

4.2. A numerical example

As sample problem a masonry wall, of dimensions $(4 \times 4)m$, made of the same orthotropic blocks texture of Section 3 was analysed. The wall, fixed on its basis, was subjected to a vertical body force, the self-weight, and to a monotonically increasing horizontal body force, proportional to the vertical force through the factor λ. This problem can be interpreted as a simplified modelling of the seismic actions through static forces.

In this case, considering the orthonormal basis $\{\mathbf{m}_p, \mathbf{n}_p\}$ for the pth contact surface between blocks, the non-null constitutive constants for the micromodel are:

$$(\mathbf{K}_p)_{mm} = \mathbf{K}_p \cdot \mathbf{m}_p \otimes \mathbf{m}_p = 1.50 \cdot 10^3 \text{ MN/m}; \quad (\mathbf{K}_p)_{nn} = \mathbf{K}_p\mathbf{n}_p \otimes \mathbf{n}_p = 3.00 \cdot 10^3 \text{ MN/m},$$

$$(\mathsf{K}_p)_{mnmn} = \mathsf{K}_p \cdot \mathbf{m}_p \otimes \mathbf{n}_p \otimes \mathbf{m}_p \otimes \mathbf{n}_p = 3.00 \cdot 10 \text{ MNm}.$$

and, from equations (14), the non-null elasticities of the macromodel are:

$$(\mathsf{A})_{1111} = \mathsf{A} \cdot \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 = 10.00 \cdot 10^3 \text{ MN/m}^2,$$

$$(\mathsf{A})_{2222} = \mathsf{A} \cdot \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 = 4.00 \cdot 10^3 \text{ MN/m}^2,$$

$$(\mathsf{A})_{1212} = \mathsf{A} \cdot \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 = 2.00 \cdot 10^3 \text{ MN/m}^2,$$

$$(\mathsf{A})_{2121} = \mathsf{A} \cdot \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 = 8.00 \cdot 10^3 \text{ MN/m}^2,$$

$$(\mathsf{C})_{121121} = \mathsf{C} \cdot \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 = 1.20 \cdot 10^2 \text{ MN},$$

$$(\mathsf{C})_{122122} = \mathsf{C} \cdot \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 = 0.40 \cdot 10^2 \text{ MN}.$$

The tensile strength at the interfaces and the friction angle are assumed constant: $a_p = 1 \cdot 10^{-2}$ MN; $\phi_p = 0.6$.

The micropolar solution was performed using the described F.E. algorithm (Section 4.1.) considering the yield conditions (20) that in the two-dimensional frame reduce to

$$\begin{aligned} &\left|(\nabla\mathbf{u} - \mathbf{W}) \cdot \mathbf{K}_p\mathbf{m}_p \otimes (\mathbf{a} - \mathbf{b})\right| \leqslant \tan\phi_p(a_p - (\nabla\mathbf{u} - \mathbf{W}) \cdot \mathbf{K}_p\mathbf{n}_p \otimes (\mathbf{a} - \mathbf{b})), \\ &(\nabla\mathbf{u} - \mathbf{W}) \cdot \mathbf{K}_p\mathbf{n}_p \otimes (\mathbf{a} - \mathbf{b}) \leqslant a_p, \\ &\left|\nabla\mathbf{W} \cdot \mathsf{K}_p\mathbf{m}_p\mathbf{n}_p \otimes (\mathbf{a} - \mathbf{b})\right| \leqslant d_p(a - (\nabla\mathbf{u} - \mathbf{W}) \cdot \mathbf{K}_p\mathbf{n}_p \otimes (\mathbf{a} - \mathbf{b})), \end{aligned} \tag{21}$$

where $d_p = 2 \cdot 10^{-1}$ m is the constant length of the contact surfaces. By increasing the load factor, λ, till the value corresponding to the last balanced solution, the collapse multiplier obtained was $\lambda_c = 0.339$.

In order to make a qualitative comparison, the solution for the discrete model was also directly obtained using the algorithm presented in (Baggio and Trovalusci, 2000). This algorithm performs non-standard lim-

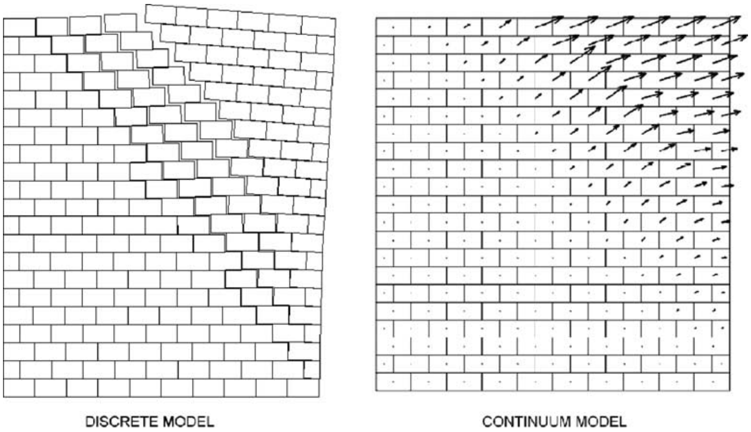

Fig. 4. Wall with body forces, non-linear solution. Deformed shape and displacement vectors of the blocks' centres evaluated by interpolation.

it analysis of rigid blocks' systems interacting through no-tension and frictional interfaces solving a non-linear and non-convex mathematical programming problem. Fig. 4 shows on the left side the collapse mechanism obtained for $\lambda_c = 0.354$.

In the right side of Fig. 4 the displacement fields of the micropolar FEM solution is represented by means of the displacements vectors of the centres of the blocks, evaluated by interpolation of the found nodal displacements.

In order to have also a quantitative comparison, as for the linear examples examined, the discrete problem was also solved using the computer code presented in the work (Trovalusci, 1992) considering the system of rigid blocks interacting through non-linear longitudinal, transversal and rotational springs. In this case the found collapse load multiplier, corresponding to the last balanced solution, is $\lambda_c = 0.341$.

In Fig. 5, on the top size, the contour lines of the norm of the displacement fields for the discrete model $|\mathbf{u}^a|$, left, and for the continuum model $|\mathbf{u}|$, right, are reported. On the bottom side the contour lines of the rotations of the blocks $(\mathbf{W}^a)_{12}$, left, and of the microrotation $(\mathbf{W})_{12}$, right, are shown.

Also in the non-linear framework, the results obtained for the Cosserat macromodel are in good agreement with the results obtained for the discrete micromodel. All these results can finally be compared with the experimental result reported in (Trovalusci and Masiani, 2003), for which the collapse value is $\lambda_c = 0.35$.

5. Final remarks

The macroscopic behaviour of materials with heterogeneities, regularly disposed and significant in size, can be studied both at the macro and microscales.

A multiscale approach, based on the assumptions of the molecular theory of elasticity, has been proposed in order to model the macroscale behaviour of such materials retaining memory of the features exhibited at finer scales. Following this approach, the constitutive model for an equivalent homogeneous continuum is "naturally" defined resulting, in most of the cases, in a multifield model with non-standard strain and stress measures always thermodynamically consistent.

For example, for the blocky materials studied in this paper, the equivalent macromodel results to be a micropolar continuum with response functions completely defined from the characteristics of the

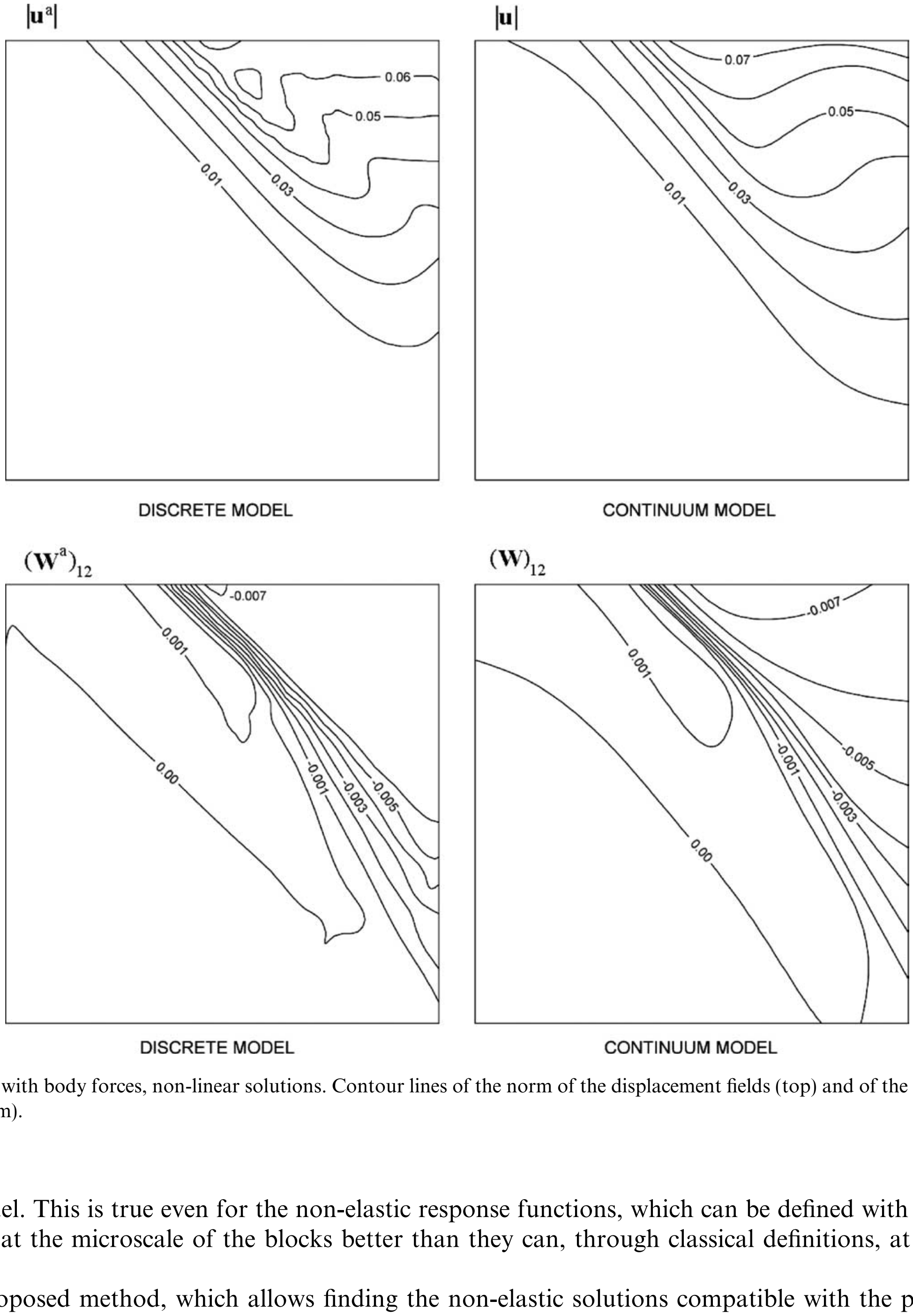

Fig. 5. Wall with body forces, non-linear solutions. Contour lines of the norm of the displacement fields (top) and of the local rotation fields (bottom).

micromodel. This is true even for the non-elastic response functions, which can be defined with a coherent approach at the microscale of the blocks better than they can, through classical definitions, at the macro scale.

The proposed method, which allows finding the non-elastic solutions compatible with the proper constraints for the constitutive functions at the microscale, has the advantage to take into account non-elastic behaviours, such as no-tension and frictional, with a clear mechanical meaning.

Numerical comparisons with both non-standard limit analyses and experimental tests confirm these results.

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